Density of integral points on algebraic varieties

Brendan Hassett and Yuri Tschinkel

February 1, 2008

1 Introduction

Let K be a number field, S a finite set of valuations of K, including the archimedean valuations, and $\mathcal{O}_S$ the ring of S-integers. Let X be an algebraic variety defined over K and D a divisor on X. We will use $\mathcal{X}$ and $\mathcal{D}$ to denote models over $\mathrm{Spec}(\mathcal{O}_S)$.

We will say that integral points on (X, D) (see Section 2 for a precise definition) are potentially dense if they are Zariski dense on some model $(\mathcal{X}, \mathcal{D})$, after a finite extension of the ground field and after enlarging S. A central problem in arithmetic geometry is to find conditions insuring potential density (or nondensity) of integral points. This question motivates many interesting and concrete problems in classical number theory, transcendence theory and algebraic geometry, some of which will be presented below.

If we think about general reasons for the density of points - the first idea would be to look for the presence of a large automorphism group. There are many beautiful examples both for rational and integral points, like K3 surfaces given by a bihomogeneous $(2, 2, 2)$ form in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ or the classical Markov equation $x^2 + y^2 + z^2 = 3xyz$. However, large automorphism groups are "sporadic" - they are hard to find and usually, they are not well behaved in families. There is one notable exception - namely automorphisms of algebraic groups, like tori and abelian varieties.

Thus it is not a surprise that the main geometric reason for the abundance of rational points on varieties treated in the recent papers [10], [3], [11] is the presence of elliptic or, more generally, abelian fibrations with *multisections* having a dense set of rational points and subject to some *nondegeneracy* conditions. Most of the effort goes into ensuring these conditions.

In this paper we focus on cases when D is nonempty. We give a systematic treatment of known approaches to potential density and present several new ideas for proofs. The analogs of elliptic fibrations in log geometry are conic bundles with a bisection removed. We develop the necessary techniques to translate the presence of such structures to statements about density of integral points and give a number of applications.

The paper is organized as follows: in Section 2 we introduce the main definitions and notations. Section 3 is geometrical - we introduce the relevant concepts from the log minimal model program and formulate several geometric problems inspired by questions about integral points. In Section 4, we recall the fibration method and nondegeneracy properties of multisections. We consider approximation methods in Section 5. Section 6 is devoted to the study of integral points on conic bundles with sections and bisections removed. In the final section, we survey the known results concerning potential density for integral point on log K3 surfaces.

Acknowledgements. The first author was partially supported by an NSF Postdoctoral Research Fellowship. The second author was partially supported by the NSA. We benefitted from conversations with Y. André, F. Bogomolov, A. Chambert-Loir, J.-L. Colliot-Thélène, J. Kollár, D. McKinnon, and B. Mazur. We are grateful to P. Vojta for comments that improved the paper, especially Proposition 3.12, and to D.W. Masser for information on specialization of nondegenerate sections. Our approach in Section 6 is inspired by the work of F. Beukers (see [1] and [2]).

2 Generalities

2.1 Integral points

Let $\pi : \mathcal{U} \to \mathrm{Spec}(\mathcal{O}_S)$ be a flat scheme over $\mathcal{O}_S$ with generic fiber U. An integral point on $\mathcal{U}$ is a section of π; the set of such points is denoted $\mathcal{U}(\mathcal{O}_S)$.

In the sequel, $\mathcal{U}$ will be the complement to a reduced effective Weil divisor $\mathcal{D}$ in a normal proper scheme $\mathcal{X}$, both generally flat over $\mathrm{Spec}(\mathcal{O}_S)$. Hence an S-integral point P of $(\mathcal{X}, \mathcal{D})$ is a section $s_P : \mathrm{Spec}(\mathcal{O}_S) \to \mathcal{X}$ of π, which does not intersect $\mathcal{D}$, that is, for each prime ideal $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_S)$ we have $s_P(\mathfrak{p}) \notin \mathcal{D}_{\mathfrak{p}}$. We denote by X (resp. D) the corresponding generic fiber. We generally assume that X is a variety (i.e., a geometrically integral scheme);

frequently X is smooth and D is normal crossings. Potential density of integral points on $(\mathcal{X}, \mathcal{D})$ does not depend on the choice of S or on the choices of models over $\mathrm{Spec}(\mathcal{O}_S)$, so we will not always specify them. Hopefully, this will not create any confusion.

If D is empty then every K-rational point of X is an S-integral point for $(\mathcal{X}, \mathcal{D})$ (on some model). Every K-rational point of X, not contained in D is S-integral on $(\mathcal{X}, \mathcal{D})$ for S large enough. Clearly, for any $\mathcal{X}$ and $\mathcal{D}$ there exists a finite extension K'/K and a finite set S' of prime ideals in $\mathcal{O}_{K'}$ such that there is an S'-integral point on $(\mathcal{X}', \mathcal{D}')$ (where $\mathcal{X}'$ is the basechange of $\mathcal{X}$ to $\mathrm{Spec}(\mathcal{O}'_S)$).

The definition of integral points can be generalized as follows: let $\mathcal{Z}$ be any subscheme of $\mathcal{X}$, flat over $\mathcal{O}_S$. An S-integral point for $(\mathcal{X}, \mathcal{Z})$ is an $\mathcal{O}_S$-valued point of $\mathcal{X} \setminus \mathcal{Z}$.

2.2 Vojta's conjecture

A *pair* consists of a proper normal variety X and a reduced effective Weil divisor $D \subset X$. A *morphism of pairs* $\varphi : (X_1, D_1) \to (X_2, D_2)$ is a morphism $\varphi : X_1 \to X_2$ such that $\varphi^{-1}(D_2)$ is a subset of D_1. In particular, φ restricts to a morphism $X_1 \setminus D_1 \to X_2 \setminus D_2$. A morphism of pairs is *dominant* if $\varphi : X_1 \to X_2$ is dominant. If (X_1, D_1) dominates (X_2, D_2) then integral points are dense on (X_2, D_2) when they are dense on (X_1, D_1) (after choosing appropriate integral models.) A morphism of pairs is *proper* if $\varphi : X_1 \to X_2$ is proper and the restriction $X_1 \setminus D_1 \to X_2 \setminus D_2$ is also proper; equivalently, we may assume that $\varphi : X_1 \to X_2$ is proper and D_1 is a subset of $\varphi^{-1}(D_2)$. A *resolution* of the pair (X, D) is a proper morphism of pairs $\rho : (\tilde{X}, \tilde{D}) \to (X, D)$ such that $\rho : \tilde{X} \to X$ is birational, $\tilde{X}$ is smooth, and $\tilde{D}$ is normal crossings.

Let X be a normal proper variety of dimension d. Recall that a Cartier divisor $D \subset X$ is *big* if $h^0(\mathcal{O}_X(nD)) > Cn^d$ for some $C > 0$ and all n sufficiently large and divisible.

Definition 2.1 A pair (X, D) is of log general type if it admits a resolution $\rho : (\tilde{X}, \tilde{D}) \to (X, D)$ with $\omega_{\tilde{X}}(\tilde{D})$ big.

Let us remark that the definition does not depend on the resolution.

Conjecture 2.2 *(Vojta, [29]) Let (X, D) be a pair of log general type. Then integral points on (X, D) are not potentially dense.*

This conjecture is known when X is a semiabelian variety ([9], [30], [15]). Vojta's conjecture implies that a pair with dense integral points cannot dominate a pair of log general type.

We are interested in geometric conditions which would insure potential density of integral points. The most naive statement would be the direct converse to Vojta's conjecture. However this can't be true even when $D = \emptyset$. Indeed, varieties which are not of general type may dominate varieties of general type, or more generally, admit finite étale covers which dominate varieties of general type (see the examples in [7]). In the next section we will analyze other types of covers with the same arithmetic property.

3 Geometry

3.1 Morphisms of pairs

Definition 3.1 We will say that a class of dominant morphisms of pairs $\varphi : (X_1, D_1) \to (X_2, D_2)$ is arithmetically continuous if the density of integral points on (X_2, D_2) implies potential density of integral points on (X_1, D_1).

For example, assume that $D = \emptyset$. Then any projective bundle in the Zariski topology $\mathbb{P} \to X$ is arithmetically continuous. In the following sections we present other examples of arithmetically continuous morphisms of pairs.

Definition 3.2 A pseudo-étale cover of pairs $\varphi : (X_1, D_1) \to (X_2, D_2)$ is a proper dominant morphism of pairs such that

a) $\varphi : X_1 \to X_2$ is generically finite, and

b) the map from the normalization X_2^{norm} of X_2 (in the function field of X_1) onto X_2 is étale away from D_2.

Remark 3.3 For every pair (X, D) there exists a birational pseudo-étale morphism $\varphi : (\tilde{X}, \tilde{D}) \to (X, D)$ such that $\tilde{X}$ is smooth and $\tilde{D}$ is normal crossings.

The following theorem is a formal generalization of the well-known theorem of Chevalley-Weil. It shows that potential density is stable under pseudo-étale covers of pairs.

Theorem 3.4 *Let $\varphi : (X_1, D_1) \to (X_2, D_2)$ be a pseudo-étale cover of pairs. Then φ is arithmetically continuous.*

Remark 3.5 An elliptic fibration $E \to X$, isotrivial on $X \setminus D$, is arithmetically continuous. Indeed, it splits after a pseudo-étale morphism of pairs and we can apply Theorem 3.4.

The following example is an integral analog of the example of Skorobogatov, Colliot-Thélène and Swinnerton-Dyer ([7]) of a variety which does not dominate a variety of general type but admits an étale cover which does.

Example 3.6 Consider $\mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $(x_1, y_1), (x_2, y_2)$ and involutions

$$j_1(x_1, y_1) = (-x_1, y_1) \quad j_2(x_2, y_2) = (y_2, x_2)$$

on the factors. Let j be the induced involution on the product; it has fixed points

$$\begin{array}{ll} x_1 = 0 & x_2 = y_2 \\ x_1 = 0 & x_2 = -y_2 \\ y_1 = 0 & x_2 = y_2 \\ y_1 = 0 & x_2 = -y_2 \end{array} .$$

The first projection induces a map of quotients

$$(\mathbb{P}^1 \times \mathbb{P}^1) / \langle j \rangle \to \mathbb{P}^1 / \langle j_1 \rangle .$$

We use X to denote the source; the target is just $\mathrm{Proj}(\mathbb{C}[x_1^2, y_1]) \simeq \mathbb{P}^1$. Hence we obtain a fibration $f : X \to \mathbb{P}^1$. Note that f has two nonreduced fibers, corresponding to $x_1 = 0$ and $y_1 = 0$ respectively. Let D be the image in X of

$$(x_1 = 0) \cup (y_1 = 0) \cup (x_2 = m_2 y_2) \cup (x_2 = m_1 y_2)$$

(where m_1, m_2 are nonzero and distinct). Since D intersects the general fiber of f in just two points, (X, D) is not of log general type.

We can represent X as a degenerate quartic Del Pezzo surface with four A1 singularities (see figure 1). If we fix invariants

$$a = x_1^2 x_2 y_2,\ b = x_1^2(x_2^2 + y_2^2),\ c = x_1 y_1 (x_2^2 - y_2^2),\ d = y_1^2(x_2^2 + y_2^2),\ e = y_1^2 x_2 y_2$$

then X is given as a complete intersection of two quadrics:

$$ad = be, \qquad c^2 = bd - 4ae.$$

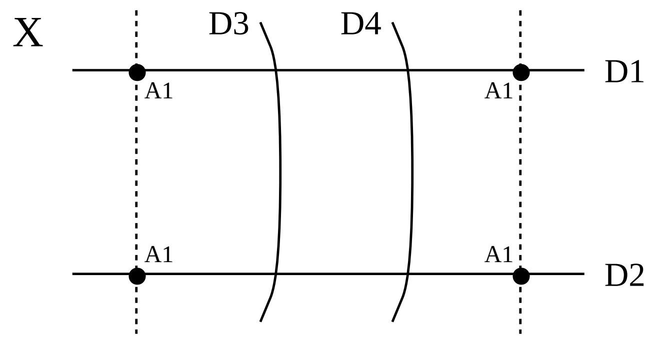

Figure 1: The log surface (X, D)

The components of D satisfy the equations

$$\begin{aligned} D_1 &= \{a = b = c = 0\} \\ D_2 &= \{c = d = e = 0\} \\ D_3 &= \{(1 + m_1^2)a - m_1 b = (1 + m_1^2)e - m_1 d = 0\} \\ D_4 &= \{(1 + m_2^2)a - m_2 b = (1 + m_2^2)e - m_2 d = 0\}. \end{aligned}$$

We claim that (X, D) does not admit a dominant map onto a variety of log general type and that there exists a pseudo-étale cover of (X, D) which does. Indeed, the preimage of $X \setminus D$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is

$$(\mathbb{A}^1 \setminus 0) \times (\mathbb{P}^1 \setminus \{m_1, m_2, 1/m_1, 1/m_2\}),$$

which dominates a curve of log general type, namely, $\mathbb{P}^1$ minus four points. However, (X, D) itself cannot dominate a curve of log general type. Any such curve must be rational, with at least three points removed; however, the boundary D contains at most two mutually disjoint irreducible components.

The following was put forward as a possible converse to Vojta's conjecture.

Problem 3.7 (Strong converse to Vojta's conjecture) Assume that the pair (X_2, D_2) does not admit a pseudo-étale cover $(X_1, D_1) \to (X_2, D_2)$ such that (X_1, D_1) dominates a pair of log general type. Are integral points for (X_2, D_2) potentially dense?

3.2 Projective bundles in the étale topology

We would like to produce further classes of dominant arithmetically continuous morphisms $(X_1, D_1) \to (X_2, D_2)$.

Theorem 3.8 *Let $\varphi : (X_1, D_1) \to (X_2, D_2)$ be a projective morphism of pairs such that φ is a projective bundle (in the étale topology) over $X_2 \setminus D_2$. We also assume that $\varphi^{-1}(D_2) = D_1$. Then φ is arithmetically continuous.*

Proof. We are very grateful to Prof. Colliot-Thélène for suggesting this proof.

Choose models $(\mathcal{X}_i, \mathcal{D}_i)$ $(i = 1, 2)$ over some ring of integers $\mathcal{O}_S$, so that the morphism φ is well-defined and satisfies our hypotheses. (We enlarge S as necessary.)

We recall basic properties of the Brauer group $\mathrm{Br}(\mathcal{O}_S)$. Let v denote a place for the quotient field K and K_v the corresponding completion. Class-field theory gives the following exact sequence

$$0 \to \mathrm{Br}(\mathcal{O}_S) \to \mathrm{Br}(K) \to \oplus_{v \notin S} \mathrm{Br}(K_v).$$

The Brauer groups of the local fields corresponding to nonarchimedean valuations are isomorphic to $\mathbb{Q}/\mathbb{Z}$. Given a finite extension of K_w/K_v of degree n the induced map on Brauer groups is multiplication by n.

Each $\mathcal{O}_S$-integral point of (X_2, D_2) yields an element of $\mathrm{Br}(\mathcal{O}_S)$ of order r. This gives elements of $\mathrm{Br}(K_v)$ which are zero unless $v \in S$. It suffices to find an extension K'/K inducing a cyclic extension of K_v of order divisible by r for all $v \in S$. Indeed, such an extension necessarily kills any element of $\mathrm{Br}(\mathcal{O}_S)$ of order r. $\square$

Remark 3.9 Let X be a smooth simply connected projective variety which does not dominate a variety of general type. It may admit an projective bundle (in the étale topology) $\varphi : \mathbb{P} \to X$, for example if X is a K3 surface. However, $\mathbb{P}$ cannot dominate a variety of general type. Indeed, given a dominant morphism $\pi : \mathbb{P} \to Y$, the fibers of φ are mapped to points by π. In particular, π necessarily factors through φ. (We are grateful to J. Kollár for emphasizing this point.)

Problem 3.10 (Geometric counterexamples to Problem 3.7) Are there pairs which do not admit pseudo-étale covers dominating pairs of log general type but which do admit arithmetically continuous covers dominating pairs of log general type?

3.3 Punctured varieties

In Section 3.1 we have seen that potential density of integral points is preserved under pseudo-étale covers. It is not an easy task, in general, to check whether or not some given variety (like an elliptic surface) admits a (pseudo-) étale cover dominating a variety of general type. What happens if we modify the variety (or pair) without changing the fundamental group?

Problem 3.11 (Geometric puncturing problem) Let X be a projective variety with canonical singularities and Z a subvariety of codimension ≥ 2. Assume that no (pseudo-) étale cover of $(X, \emptyset)$ dominates a variety of general type. Then (X, Z) admits no pseudo-étale covers dominating a pair of log general type. A weaker version would be to assume that X and Z are smooth.

By definition, a pseudo-étale cover of (X, Z) is a pseudo-étale cover of a pair (X', D'), where X' is proper over X and $X' \setminus D' \simeq X \setminus Z$.

Proposition 3.12 *Assume X and Z are as in Problem 3.11, and that X is smooth. Then*

a) No pseudo-étale covers of (X, Z) dominate a curve of log general type.

b) No pseudo-étale covers of (X, Z) dominate a variety of log general type of the same dimension.

Proof. Suppose we have a pseudo-étale cover $\rho : (X_1, D_1) \to (X, Z)$ and a dominant morphism $\varphi : (X_1, D_1) \to (X_2, D_2)$ to a variety of log general type. By Remark 3.3, we may take the X_i smooth and the D_i normal crossings. Since D_1 is exceptional with respect to ρ, Iitaka's Covering Theorem ([12] Theorem 10.5) yields an equality of Kodaira dimensions

$$\kappa(K_X) = \kappa(K_{X_1} + D_1).$$

Assume first that X_2 is a curve. We claim it has genus zero or one. Let X^{norm} be the normalization of X in the function field of X_1. The induced morphism $g : X^{\text{norm}} \to X$ is finite, surjective, and branched only over Z, a codimension ≥ 2 subset of X. Since X is smooth, it follows that g is étale. If X_2 has genus ≥ 2 then $\varphi : X_1 \to X_2$ is constant along the fibers of $X_1 \to X^{\text{norm}}$, and thus descends to a map $X^{\text{norm}} \to X_2$. This would contradict our assumption that no étale cover of X dominates a variety of general type.

Choose a point $p \in D_2$ and consider the divisor $F = \varphi^{-1}p$. Note that $2F$ moves because $2p$ moves on X_2. However, $2F$ is supported in D_1, which lies in the exceptional locus for ρ, and we obtain a contradiction.

Now assume φ is generically finite. We apply the Logarithmic Ramification Formula to φ (see [12] Theorem 11.5)

$$K_{X_1} + D_1 = \varphi^*(K_{X_2} + D_2) + R$$

where R is the (effective) logarithmic ramification divisor. Applying the Covering Theorem again, we find that $\kappa(K_{X_1}+D_1-R) = \kappa(K_{X_2}+D_2) = \dim(X)$. It follows that $K_{X_1} + D_1$ is also big, which contradicts the assumption that X is not of general type. $\square$

Problem 3.13 (Arithmetic puncturing problem) Let X be a projective variety with canonical singularities and Z a subvariety of codimension ≥ 2. Assume that rational points on X are potentially dense. Are integral points on (X, Z) potentially dense?

For simplicity, one might first assume that X and Z are smooth. Note that some conditions on the singularities of X are necessary. For example, blow up $\mathbb{P}^2$ in 20 points lying along a smooth quartic curve C. Assume that the divisor class of the points equals $5H$, where H is the hyperplane class of C. Then the linear series of quintics containing the 20 points gives a birational map contracting C. Let X be the resulting surface and Z the singular point. Rational points on X are dense but density of integral points on (X, Z) would contradict Vojta's conjecture.

Remark 3.14 Assume that Problem 3.13 has a positive solution. Then potential density of rational points holds for all K3 surfaces.

Indeed, if Y is a K3 surface of degree $2n$ then potential density of rational points holds for the symmetric product $X = Y^{(n)}$ (see [11]). Denote by Z the large diagonal in X and by Δ the large diagonal in Y^n (the ordinary product). Assume that integral points on (X, Z) are potentially dense. Then, by Theorem 3.4 integral points on (Y^n, Δ) are potentially dense. This implies potential density for rational points on Y.

4 The fibration method and nondegenerate multisections

This section is included as motivation. Let B be an algebraic variety, defined over a number field K and $\pi : G \to B$ be a group scheme over B. We will be mostly interested in the case when the generic fiber is an abelian variety or a split torus $\mathbb{G}_m^n$. Let s be a section of π. Shrinking the base we may assume that all fibers of G are smooth. We will say that s is *nondegenerate* if $\cup_n s^n$ is Zariski dense in G.

Problem 4.1 (Specialization) Assume that $G \to B$ has a nondegenerate section s. Describe the set of $b \in B(K)$ such that $s(b)$ is nondegenerate in the fiber G_b.

For simple abelian varieties over a field a point of infinite order is nondegenerate. If $E \to B$ is a Jacobian elliptic fibration with a section s of infinite order then this section is automatically nondegenerate, and $s(b)$ is nondegenerate if it is nontorsion. By a result of Néron (see [25] 11.1), the set of $b \in B(K)$ such that $s(b)$ is not of infinite order is *thin*; this holds true for abelian fibrations of arbitrary dimension.

For abelian fibrations $A \to B$ with higher-dimensional fibers, one must also understand how rings of endomorphisms specialize. The set of $b \in B(K)$ for which the restriction

$$\mathrm{End}(A) \to \mathrm{End}(A(b))$$

fails to be surjective is also thin; this is a result of Noot [20] Corollary 1.5. In particular, a nondegenerate section of a family of generically simple abelian varieties specializes to a nondegenerate point outside a thin set of fibers.

More generally, given an arbitrary abelian fibration $A \to B$ and a nondegenerate section s, the set of $b \in B(K)$ such that $s(b)$ is degenerate is thin in B. (We are grateful to Masser for pointing out the proof.) After replacing A by an isogenous abelian variety and taking a finite extension of the function field $K(B)$, we obtain a family $A' \to B'$ with $A' \simeq A_1^{r_1} \times \ldots \times A_m^{r_m}$, where the A_j are (geometrically) simple and mutually non-isogenous. By the Theorems of Néron and Noot, the $A_j(b')$ are simple and mutually non-isogenous away from some thin subset of B'. A section s' of $A' \to B'$ is nondegenerate iff its projection onto each factor $A_j^{r_j}$ is nondegenerate; for b' not contained in our

thin subset, $s'(b')$ is nondegenerate iff its projection onto each $A_j^{r_j}(b')$ is nondegenerate. Hence we are reduced to proving the claim for each $A_j^{r_j}$. Since A_j is simple, a section s_j of $A_j^{r_j}$ is nondegenerate iff its projections $s_{j,1}, \ldots, s_{j,r_j}$ are linearly independent over $\mathrm{End}(A_j)$. Away from a thin subset of B', the same statement holds for the specializations to b'. However, Néron's theorem implies that $s_{j,1}(b'), \ldots, s_{j,r_j}(b')$ are linearly independent away from a thin subset.

Remark 4.2 There are more precise versions of Néron's Theorem due to Demyanenko, Manin and Silverman (see [27], for example). Masser has proposed another notion of what it means for a subset of $B(K)$ to be small, known as 'sparcity'. For instance, the endomorphism ring of a family of abelian varieties changes only on a 'sparse' set of rational points of the base (see [17]). For an analogue to Néron's Theorem, see [16].

Similar results hold for algebraic tori and are proved using a version of Néron's Theorem for $\mathbb{G}_m^n$-fibrations (see [25] pp. 154). A sharper result (for 1-dimensional bases B) can be obtained from the following recent theorem:

Theorem 4.3 *([4]) Let C be an absolutely irreducible curve defined over a number field K and $x_1, ..., x_r$ rational functions in $K(C)$, multiplicatively independent modulo constants. Then the set of algebraic points $p \in C(\overline{\mathbb{Q}})$ such that $x_1(p), ..., x_r(p)$ are multiplicatively dependent has bounded height.*

The main idea of the papers [10], [3], [11] can be summarized as follows. We work over a number field K and we assume that all geometric data are defined over K. Let $\pi : E \to B$ be a Jacobian elliptic fibration over a one dimensional base B. This means that we have a family of curves of genus 1 and a global zero section so that every fiber is in fact an elliptic curve. Suppose that we have another section s which is of infinite order in the Mordell-Weil group of $E(K(B))$. The specialization results mentioned above show that for a Zariski dense set of $b \in B(K)$ the restriction $s(b)$ is of infinite order in the corresponding fiber E_b. If K-rational points on B are Zariski dense then rational points on E are Zariski dense as well.

Let us consider a situation when E does not have any sections but instead has a multisection M. By definition, a multisection (resp. rational multisection) M is irreducible and the induced map $M \to B$ is finite flat (resp. generically finite) of degree $\deg(M)$. The base-changed family

$E \times_B M \to M$ has the identity section Id (i.e., the image of the diagonal under $M \times_B M \to E \times_B M$) and a (rational) section

$$\tau_M := \deg(M)\mathrm{Id} - \mathrm{Tr}(M \times_B M)$$

where $\mathrm{Tr}(M \times_B M)$ is obtained (over the generic point) by summing all the points of $M \times_B M$. By definition, M is nondegenerate if τ_M is nondegenerate.

When we are concerned only with rational points, we will ignore the distinction between multisections and rational multisections, as every rational multisection is a multisection over an open subset of the base. However, this distinction is crucial when integral points are considered.

If M is nondegenerate and if rational points on M are Zariski dense then rational points on E are Zariski dense (see [3]).

Example 4.4 ([10]) Let X be a quartic surface in $\mathbb{P}^3$ containing a line L. Consider planes $\mathbb{P}^2$ passing through this line. The residual curve has degree 3. Thus we obtain an elliptic fibration on X together with the trisection L. If L is ramified in a smooth fiber of this fibration then the multisection is nondegenerate and rational points are Zariski dense.

This argument generalizes to abelian fibrations $\pi : A \to B$. However, we do not know of any simple geometric conditions insuring nondegeneracy of a (multi)section in this case. We do know that for any abelian variety A over K there exists a finite extension K'/K with a nondegenerate point in $A(K')$ (see [11]). This allows us to produce nondegenerate sections over function fields.

Proposition 4.5 *Let Y be a Fano threefold of type W_2, that is a double cover of $\mathbb{P}^3$ ramified in a smooth surface of degree 6. Then rational points on the symmetric square $Y^{(2)}$ are potentially dense.*

Proof. Observe that the symmetric square $Y^{(2)}$ is birational to an abelian surface fibration over the Grassmannian of lines in $\mathbb{P}^3$. This fibration is visualized as follows: consider two generic points in Y. Their images in $\mathbb{P}^3$ determine a line, which intersects the ramification locus in 6 points and lifts to a (hyperelliptic) genus two curve on Y. On $Y^{(2)}$ we have an abelian surface fibration corresponding to the degree 2 component of the relative Picard scheme. Now we need to produce a nondegenerate multisection. Pick two general points b_1 and b_2 on the branch surface. The preimages in Y of the corresponding tangent planes are K3 surfaces Σ_1 and Σ_2, of degree two

with ordinary double points at the points of tangency. The surfaces Σ_1 and Σ_2 therefore have potentially dense rational points (this was proved in [3]), as does $\Sigma_1 \times \Sigma_2$. This is our multisection; we claim it is nondegenerate for generic b_1 and b_2. Indeed, it suffices to show that given a (generic) point in $Y^{(2)}$, there exist b_1 and b_2 so that $\Sigma_1 \times \Sigma_2$ contains the point. Observe that through a (generic) point of $\mathbb{P}^3$, there pass many tangent planes to the branch surface. $\square$

Remark 4.6 Combining the above Proposition with the strong form of Problem 3.13 we obtain potential density of rational points on a Fano threefold of type W_2 - the last family of smooth Fano threefolds for which potential density is not known.

Here is a formulation of the fibration method useful for the analysis of integral points:

Proposition 4.7 *Let B be a scheme over a number field K, $G \to B$ a flat group scheme, $T \to B$ an étale torsor for G, and $M \subset T$ a nondegenerate multisection over B. If M has potentially dense integral points then T has potentially dense integral points.*

Proof. Without loss of generality, we may assume that B is geometrically connected and smooth. The base-changed family $T \times_B M$ dominates T, so it suffices to prove density for $T \times_B M$. Note that since M is finite and flat over B, τ_M is a well-defined *section* over all of M (i.e., it is not just a rational section). Hence we may reduce to the case of a group scheme $G \to B$ with a nondegenerate section τ.

We may choose models $\mathcal{G}$ and $\mathcal{B}$ over $\mathrm{Spec}(\mathcal{O}_S)$ so that $\mathcal{G} \to \mathcal{B}$ is a group scheme with section τ. We may also assume that $\mathcal{O}_S$-integral points of τ are Zariski dense. The set of multiples τ^n of τ, each a section of $\mathcal{G} \to \mathcal{B}$, is dense in $\mathcal{G}$ by the nondegeneracy assumption. Since each has dense $\mathcal{O}_S$-integral points, it follows that $\mathcal{O}_S$-integral points are Zariski dense. $\square$

A similar argument proves the following

Proposition 4.8 *Let $\varphi : X \to \mathbb{P}^1$ be a K3 surface with elliptic fibration. Let M be a multisection over its image $\varphi(M)$, nondegenerate and contained in the smooth locus of φ. Let $F_1, \dots, F_n$ be fibers of φ and D a divisor supported in these fibers and disjoint from M. If M has potentially dense integral points then (X, D) has potentially dense integral points.*

Proof. We emphasize that X is automatically minimal and the fibers of φ are reduced (see [3]). Our assumptions imply that M is finite and flat over $\varphi(M)$.

After base-changing to M, we obtain a Jacobian elliptic fibration $X' := X \times_{\mathbb{P}^1} M$ with identity and a nondegenerate section τ_M. Let $G \subset X'$ be the open subset equal to the connected component of the identity. Since $D' := D \times_{\mathbb{P}^1} M$ is disjoint from the identity, it is disjoint from G. Hence it suffices to show that G has potentially dense integral points.

We assumed that M is contained in the smooth locus of φ, so τ_M is contained in the grouplike part of X', and some multiple of τ_M is contained in G. Repeating the argument for Proposition 4.7 gives the result. □

5 Approximation techniques

In this section we prove potential density of integral points for certain pairs (X, D) using congruence conditions to control intersections with the boundary. Several of these examples are included as support for the statement of Problem 3.13.

Proposition 5.1 *Let $G = \prod_j^N G_j$ where G_j are algebraic tori $\mathbb{G}_m$ or geometrically simple abelian varieties. Let Z be a subvariety in G of codimension $> \mu = \max_j(\dim(G_j))$ and let $U = G \setminus Z$ be the complement. Then integral points on U are potentially dense.*

Proof. We are grateful to D. McKinnon for inspiring the following argument.

The proof proceeds by induction on the number of components N. The base case $N = 1$ follows from the fact that rational points on tori and abelian varieties are potentially dense, so we proceed with the inductive step. Consider the projections $\pi' : G \to G' = \prod_{j \neq N} G_j$ and $\pi_N : G \to G_N$. By assumption, generic fibers of π' are geometrically disjoint from Z.

Choose a ring of integers $\mathcal{O}_S$ and models $\mathcal{G}_j$ over $\mathrm{Spec}(\mathcal{O}_S)$. We assume that each $\mathcal{G}_j$ is smooth over $\mathrm{Spec}(\mathcal{O}_S)$ and that $\mathcal{G}_N$ has a nondegenerate point q (see [11], for example, for a proof of the existence of such points on abelian varieties).

Let $\mathcal{T}$ be any subscheme of $\mathcal{G}_N$ supported over a finite subset of $\mathrm{Spec}(\mathcal{O}_S)$ such that $\mathcal{G}_N$ has an $\mathcal{O}_S$-integral point p_N disjoint from $\mathcal{T}$. We claim that

such integral points are Zariski dense. Indeed, for some $m > 0$ we have

$$mq \equiv 0 \pmod{\mathfrak{p}}$$

for each $\mathfrak{p} \in \mathrm{Spec}(O_S)$ over which $\mathcal{T}$ has support. Hence we may take the translations of p_N by multiples of mq.

After extending $\mathcal{O}_S$, we may assume U has at least one integral point $p = (p', p_N)$ so that ${\pi'}^{-1}(p')$ and $\pi_N^{-1}(p_N)$ intersect Z in the expected dimensions. In particular, ${\pi'}^{-1}(p')$ is disjoint from Z. By the inductive hypothesis, we may extend $\mathcal{O}_S$ so that

$$(\pi_N^{-1}(p_N) \simeq \mathcal{G}', \pi_N^{-1}(p_N) \cap \mathcal{Z})$$

has dense integral points. In particular, almost all such integral points are not contained in $\pi'(\mathcal{Z})$, a closed proper subscheme of $\mathcal{G}'$. Let r be such a point, so that $F_r = {\pi'}^{-1}(r) \simeq \mathcal{G}_N$ intersects $\mathcal{Z}$ in a subscheme $\mathcal{T}$ supported over a finite number of primes. Since $(r, p_N) \in F_r$ is disjoint from $\mathcal{T}$, the previous claim implies that the integral points of F_r disjoint from $\mathcal{T}$ are Zariski dense. As r varies, we obtain a Zariski dense set of integral points on $\mathcal{G} \setminus \mathcal{Z}$. □

Corollary 5.2 Let X be a toric variety and $Z \subset X$ a subvariety of codimension ≥ 2, defined over a number field. Then integral points on (X, Z) are potentially dense.

Another special case of the Arithmetic puncturing problem 3.13 is the following:

Problem 5.3 Are integral points on punctured simple abelian varieties of dimension $n > 1$ potentially dense?

Example 5.4 Potential density of integral points holds for simple abelian varieties punctured in the origin, provided that their ring of endomorphisms contains units of infinite order.

6 Conic bundles and integral points

Let K be a number field, S a finite set of places for K (including all the infinite places), $\mathcal{O}_S$ the corresponding ring of S-integers, and $\eta \in \mathrm{Spec}(\mathcal{O}_S)$ the

generic point. For each place v of K, let K_v be the corresponding complete field and $\mathfrak{o}_v$ the discrete valuation ring (if v is nonarchimedean). As before, we use calligraphic letters (e.g., $\mathcal{X}$) for schemes (usually flat) over $\mathcal{O}_S$ and roman letters (e.g., X) for the fiber over η.

6.1 Results on linear algebraic groups

Consider a linear algebraic group G/K. Choose a model $\mathcal{G}$ for G over $\mathcal{O}_S$, i.e., a flat group scheme of finite type $\mathcal{G}/\mathcal{O}_S$ restricting to G at the generic point. This may be obtained by fixing a representation $G \hookrightarrow \mathrm{GL}_n(K)$ (see also [31] §10-11). The S-rank of G (denoted $\mathrm{rank}(G, \mathcal{O}_S)$) is defined as the rank of the abelian group of sections of $\mathcal{G}(\mathcal{O}_S)$ over $\mathcal{O}_S$. This does not depend on the choice of a model. Indeed, consider two models $\mathcal{G}_1$ and $\mathcal{G}_2$ with a birational map $b : \mathcal{G}_1 \dashrightarrow \mathcal{G}_2$; of course, b is trivial over the generic point and the proper transform of the identity section I_1 is the identity. There is a subscheme $Z \subset \mathrm{Spec}(\mathcal{O}_S)$ with finite support such that the indeterminacy of b is in the preimage of Z. It follows that the sections of $\mathcal{G}_1$ congruent to I_1 modulo Z have proper transforms which are sections of $\mathcal{G}_2$. Such sections form a finite-index subgroup of $\mathcal{G}_1(\mathcal{O}_S)$.

Let $\mathbb{G}_m$ be the multiplicative group over $\mathbb{Z}$, i.e., $\mathrm{Spec}(\mathbb{Z}[x,y]/\langle xy-1\rangle)$; it can be defined over an arbitrary scheme by extension of scalars. There is a natural projection

$$\mathbb{G}_m(\mathbb{Z}) \to \mathrm{Spec}(\mathbb{Z}[x]) = \mathbb{A}^1_{\mathbb{Z}} \subset \mathbb{P}^1_{\mathbb{Z}}$$

so that $\mathbb{P}^1_{\mathbb{Z}} \setminus \mathbb{G}_m(\mathbb{Z}) = \{0, \infty\}$. A *form* of $\mathbb{G}_m$ over K is a group scheme G/K for which there exists a finite field extension K'/K and an isomorphism $G \times_K K' \simeq \mathbb{G}_m(K')$. These are classified as follows (see [22] for a complete account). Any group automorphism

$$\alpha : \mathbb{G}_m(K') \to \mathbb{G}_m(K')$$

is either inversion or the identity, depending on whether it exchanges 0 and ∞. The corresponding automorphism group is smooth, so we may work in the étale topology (see [18] Theorem 3.9). In particular,

$$K-\text{forms of } \mathbb{G}_m \simeq H^1_{\acute{e}t}(\mathrm{Spec}(K), \mathbb{Z}/2\mathbb{Z}).$$

Each such form admits a natural open imbedding into a projective curve $G \hookrightarrow X$, generalizing the imbedding of $\mathbb{G}_m$ into $\mathbb{P}^1$. The complement $D =$

$X \setminus G$ consists of two points. The Galois action on D is given by the cocycle in $H^1_{ét}(\mathrm{Spec}(K), \mathbb{Z}/2\mathbb{Z})$ classifying G.

There is a general formula for the rank due to T. Ono and J.M. Shyr (see [21], Theorem 6 and [26]). Let T_v denote the completion of T at some place v, $\hat{T}$ and $\hat{T}_v$ the corresponding character groups, and $\rho(T)$ (resp. $\rho(\mathcal{T}_v)$) the number of independent elements of $\hat{T}$ (resp. $\hat{\mathcal{T}}_v$). The formula takes the form

$$\mathrm{rank}(T, \mathcal{O}_S) = \sum_{v \in S} \rho(\mathcal{T}_v) - \rho(T).$$

For forms of $\mathbb{G}_m$ this is particularly simple. For split forms

$$\mathrm{rank}(\mathbb{G}_m, \mathcal{O}_S) = \#\{\text{places } v \in S\} - 1.$$

Now let G/K be a nonsplit form, corresponding to the quadratic extension K'/K, and S' the places of K' lying over the places of S. Then we have

$$\mathrm{rank}(G, \mathcal{O}_S) = \#\{\text{places } v \in S \text{ completely splitting in } S'\}.$$

6.2 Group actions and integral points

Throughout this subsection, $\mathcal{X}$ is a normal, geometrically connected scheme and $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_S)$ a flat projective morphism. Let $\mathcal{D} \subset \mathcal{X}$ be an effective reduced Cartier divisor. Contrary to our previous conventions, we do not assume that $\mathcal{D}$ is flat over $\mathcal{O}_S$. Assume that a linear algebraic group G acts on X so that $X \setminus D$ is a G-torsor.

Proposition 6.1 *There exists a model $\mathcal{G}$ for G such that $\mathcal{G}$ acts on $\mathcal{X}$ and stabilizes $\mathcal{D}$.*

Proof. Choose an imbedding $\mathcal{X} \hookrightarrow \mathbb{P}^n_{\mathcal{O}_S}$ and a compatible linearization $G \hookrightarrow \mathrm{GL}_{n+1}(K)$ (see [19], Ch. 1 §3). Let $\mathcal{G}' \hookrightarrow \mathrm{GL}_{n+1}(\mathcal{O}_S)$ be the resulting integral model of G, so that $\mathcal{G}'$ stabilizes the ideal of $\mathcal{X}$ and therefore acts on it. Furthermore, $\mathcal{G}'$ evidently stabilizes the irreducible components of $\mathcal{D}$ dominating $\mathcal{O}_S$. The fibral components of $\mathcal{D}$ are supported over a finite subset of $\mathrm{Spec}(\mathcal{O}_S)$. We take $\mathcal{G} \subset \mathcal{G}'$ to be the subgroup acting trivially over this subset; it has the desired properties. $\square$

Proposition 6.2 *Assume $(\mathcal{X}, \mathcal{D})$ has an $\mathcal{O}_S$-integral point and that G has positive $\mathcal{O}_S$-rank. Then $(\mathcal{X}, \mathcal{D})$ has an infinite number of $\mathcal{O}_S$-integral points.*

Proof. Consider the action of $\mathcal{G}(\mathcal{O}_S)$ on the integral point σ (which has trivial stabilizer). The orbit consists of $\mathcal{O}_S$-integral points of $(\mathcal{X}, \mathcal{D})$, an infinite collection because $\mathcal{G}$ has positive rank. $\square$

Now assume that X is a smooth rational curve. A rational section (resp. bisection) $\mathcal{D} \subset \mathcal{X}$ is a reduced effective Cartier divisor such that the generic fiber D is reduced of degree one (resp. two). Note that the open curve $X \setminus D$ is geometrically isomorphic to $\mathbb{P}^1 - \{\infty\}$ (resp. $\mathbb{P}^1 - \{0, \infty\}$), and thus is a torsor for some K-form G of $\mathbb{G}_a$ (resp. $\mathbb{G}_m$). This form is easily computed. Of course, $\mathbb{G}_a$ has no nontrivial forms. In the $\mathbb{G}_m$ case, we can regard D_η as an element of $H^1_{\acute{e}t}(\mathrm{Spec}(K), \mathbb{Z}/2\mathbb{Z})$, which gives the descent data for G.

The following result is essentially due to Beukers (see [1], Theorem 2.3):

Proposition 6.3 *Let $(\mathcal{X}, \mathcal{D}) \to \mathrm{Spec}(\mathcal{O}_S)$ be a rational curve with rational bisection and G the corresponding form of $\mathbb{G}_m$ (as described above). Assume that $(\mathcal{X}, \mathcal{D})$ has an $\mathcal{O}_S$-integral point and* $\mathrm{rank}(G, \mathcal{O}_S) > 0$. *Then $\mathcal{O}_S$-integral points of $(\mathcal{X}, \mathcal{D})$ are Zariski dense.*

Proof. This follows from Proposition 6.2. Given an $\mathcal{O}_S$-integral point σ of $(\mathcal{X}, \mathcal{D})$, the orbit $\mathcal{G}(\mathcal{O}_S)\sigma$ is infinite and thus Zariski dense. $\square$

Combining with the formula for the rank, we obtain the following:

Corollary 6.4 Let $(\mathcal{X}, \mathcal{D}) \to \mathrm{Spec}(\mathcal{O}_S)$ be a rational curve with rational bisection such that $(\mathcal{X}, \mathcal{D})$ has an $\mathcal{O}_S$-integral point. Assume that either

a) D is reducible over $\mathrm{Spec}(K)$ and $|S| > 1$; or

b) D is irreducible over $\mathrm{Spec}(K)$ and at least one place in S splits completely in $K(D)$.

Then $\mathcal{O}_S$-integral points of $(\mathcal{X}, \mathcal{D})$ are Zariski dense.

When D is a rational section we obtain a similar result (also essentially due to Beukers [1], Theorem 2.1):

Proposition 6.5 *Let $(\mathcal{X}, \mathcal{D}) \to \mathrm{Spec}(\mathcal{O}_S)$ be a rational curve with rational section such that $(\mathcal{X}, \mathcal{D})$ has an $\mathcal{O}_S$-integral point. Then $\mathcal{O}_S$-integral points of $(\mathcal{X}, \mathcal{D})$ are Zariski dense.*

6.3 v-adic geometry

For each place $v \in S$, consider the projective space $\mathbb{P}^1(K_v)$ as a manifold with respect to the topology induced by the v-adic absolute value on K_v.

For simplicity, this will be called the v-adic topology; we will use the same term for the induced subspace topology on $\mathbb{P}^1(K)$. Given an étale morphism of curves $f : U \to \mathbb{P}^1$ defined over K_v, we will say that $f(U(K_v))$ is a *basic étale open subset*. These are open in the v-adic topology, either by the open mapping theorem (in the archimedean case) or by Hensel's lemma (in the nonarchimedean case).

Let

$$\chi_f(B) := \#\{z \in \mathcal{O}_{\{v\}} : |z|_v \le B \text{ and } z \in f(U(K_v))\}$$

where B is a positive integer and

$$\mathcal{O}_{\{v\}} := \{z \in K : |z|_w \le 1 \text{ for each } w \ne v\}.$$

We would like to estimate the quantity

$$\mu_f := \liminf_{B\to\infty} \chi_f(B)/\chi_{\mathrm{Id}}(B)$$

i.e., the fraction of the integers contained in the image of the v-adic points of U.

Proposition 6.6 *Let $f : U \to \mathbb{P}^1$ be an étale morphism defined over K_v and $f_1 : C \to \mathbb{P}^1$ a finite morphism of smooth curves extending f. If there exists a point $q \in f_1^{-1}(\infty) \cap C(K_v)$ at which f_1 is unramified then $\mu_f = 1$.*

Proof. This follows from the fact that $f(U(K_v))$ is open if f is étale along U. □

As an illustrative example, we take $K = \mathbb{Q}$ and $K_v = \mathbb{R}$, so that $\mathcal{O}_{\{v\}} = \mathbb{Z}$. The set $f(U(\mathbb{R}))$ is a finite union of open intervals (r, s) with $r, s \in \mathbb{R} \cup \{\infty\}$, where the (finite) endpoints are branch points. We observe that

$$\mu_f = \begin{cases} 0 & \text{if } f(U(\mathbb{R})) \text{ is bounded;} \\ 1/2 & \text{if } \overline{f(U(\mathbb{R}))} \text{ contains a one-sided neighborhood of } \infty; \\ 1 & \text{if } \overline{f(U(\mathbb{R}))} \text{ contains a two-sided neighborhood of } \infty. \end{cases}$$

We can read off easily which alternative occurs in terms of the local behavior at infinity. Let $f_1 : C \to \mathbb{P}^1$ be a finite morphism of smooth curves extending f. If $f_1^{-1}(\infty)$ has no real points then $\mu_f = 0$. If $f_1^{-1}(\infty)$ has unramified (resp. ramified) real points then $\mu_f = 1$ (resp. $\mu_f > 0$.)

We specialize to the case of double covers:

Proposition 6.7 *Let $U \to \mathbb{P}^1$ be an étale morphism defined over K_v and $f_1 : C \to \mathbb{P}^1$ a finite morphism of smooth curves extending f. Assume that f_1 has degree two and ramifies at $q \in f_1^{-1}(\infty)$. Then $\mu_f > 0$.*

Proof. Of course, q is necessarily defined over K_v. The archimedean case follows from the previous example, so we restrict to the nonarchimedean case. Assume f_1 is given by

$$y^2 = c_n z^n + c_{n-1} z^{n-1} + \ldots + c_0,$$

where z is a coordinate for the affine line in $\mathbb{P}^1(K_v)$, $c_n \neq 0$, and the $c_i \in \mathfrak{o}_v$. Substituting $z = 1/t$ and $y = x/t^{\lceil n/2 \rceil}$, we obtain the equation at infinity

$$\begin{cases} x^2 = c_n + c_{n-1}t + \ldots + c_0 t^n & \text{for } n \text{ even} \\ x^2 = c_n t + c_{n-1}t^2 + \ldots + c_0 t^n & \text{for } n \text{ odd} \end{cases}.$$

If n is even then $f_1^{-1}(\infty)$ consists of two non-ramified points, so we may assume n odd. Then $f_1^{-1}(\infty)$ consists of one ramification point q, necessarily defined over K_v.

Write $c_n = u_0 \pi^\alpha$ and $z = u_1 \pi^{-\beta}$, where u_0 and u_1 are units and π is a uniformizer in $\mathfrak{o}_v$. (We may assume that some power π^r is contained in $\mathcal{O}_K$.) Our equation takes the form

$$y^2 \pi^{n\beta - \alpha} = u_0 u_1^n + c_{n-1} u_1^{n-1} \pi^{\beta - \alpha} + \ldots + c_0 u_1 \pi^{n\beta - \alpha}. \tag{1}$$

We review a property of the v-adic numbers, (proved in [24], Ch. XIV §4). Consider the multiplicative group

$$U^{(m)} := \{u \in \mathfrak{o}_v : u \equiv 1 \pmod{\pi^m}\}.$$

Then for m sufficiently large we have $U^{(m)} \subset K_v^2$. In particular, to determine whether a unit u is a square, it suffices to consider its representative mod π^m.

Consequently, if β is sufficiently large and has the same parity as α, then we can solve Equation 1 for $y \in K_v$ precisely when $u_0 u_1$ is a square. For example, choose any $M \in \mathcal{O}_K$ so that $M \equiv u_0 \pi^{(r-1)\beta} \pmod{\pi^{r\beta}}$ and set $z = M/\pi^{r\beta} \in \mathcal{O}_{\{v\}}$. Hence, of the $z \in \mathcal{O}_{\{v\}}$ with $|z|_v \leq B$ (with $B \gg 0$), the fraction satisfying our conditions is bounded from below. It follows that $\mu_f > 0$. □

Now let $f : U \to \mathbb{P}^1$ be an étale morphism of curves defined over K. Consider the function

$$\omega_{f,S}(B) := \#\{z \in \mathcal{O}_S : |z|_v \leq B \text{ for each } v \in S \text{ and } \alpha \in f(U(K))\}$$

and the quantity

$$\limsup_{B\to\infty} \omega_{f,\{v\}}(B)/\chi_f(B).$$

We expect that this is zero provided that f does not admit a rational section. We shall prove this is the case when f has degree two.

A key ingredient of our argument is a version of Hilbert's Irreducibility Theorem:

Proposition 6.8 *Let $f : U \to \mathbb{P}^1$ be an étale morphism of curves, defined over K and admitting no rational section. Then we have*

$$\limsup_{B\to\infty} \omega_{f,\{v\}}(B)/\chi_{\mathrm{Id}}(B) = 0.$$

Proof. We refer the reader to Serre's discussion of Hilbert's irreducibility theorem ([25], §9.6, 9.7). Essentially the same argument applies in our situation. □

Combining Propositions 6.6, 6.7, and 6.8, we obtain:

Corollary 6.9 Let $f : C \to \mathbb{P}^1$ be a finite morphism of smooth curves defined over K. Assume that f admits no rational section and that $f^{-1}(\infty)$ contains a K_v-rational point. We also assume that f has degree two. Then we have

$$\limsup_{B\to\infty} \omega_{f,\{v\}}(B)/\chi_f(B) = 0.$$

In particular, the set $\{z \in \mathcal{O}_{\{v\}} : z \in f(C(K_v)) \setminus f(C(K))\}$ is infinite.

6.4 A density theorem for surfaces

Geometric assumptions: Let $\mathcal{X}$ and $\mathcal{B}$ be flat and projective over $\mathrm{Spec}(\mathcal{O}_S)$ and $\phi : \mathcal{X} \to \mathcal{B}$ be a morphism. Let $\mathcal{L} \subset \mathcal{X}$ be a closed irreducible subscheme, $\mathcal{D} \subset \mathcal{X}$ a reduced effective Cartier divisor, and $\mathfrak{q} := \mathcal{D} \cap \mathcal{L}$. We assume the generic fibers satisfy the following: X is a geometrically connected surface, B a smooth curve, $\phi : X \to B$ a flat morphism such that the generic fiber is a rational curve with bisection. We also assume $L \simeq \mathbb{P}^1_K$, $\phi|L$ is finite, and L

meets D at a single point q, at which D is nonsingular. Write $\mathcal{X}'$ for $\mathcal{X} \times_{\mathcal{B}} \mathcal{L}$, $\mathcal{D}'$ for $\mathcal{D} \times_{\mathcal{B}} \mathcal{L}$, $\mathcal{L}'$ for the image of the diagonal in $\mathcal{X} \times_{\mathcal{B}} \mathcal{L}$ (now a section for $\phi' : \mathcal{X}' \to \mathcal{L}$), and $\mathfrak{q}'$ for $\mathcal{L}' \cap \mathcal{D}'$. Finally, if $\mathcal{C}'$ denotes the normalization of the union of the irreducible components of $\mathcal{D}'$ dominating $\mathcal{L}$, we assume that $\mathcal{C}' \to \mathcal{L}$ has no rational section over K (i.e., that $\mathcal{C}'$ is irreducible over K).

Arithmetic assumptions: We assume that $(\mathcal{L}, \mathfrak{q})$ has an $\mathcal{O}_S$-integral point. Furthermore, we assume that for some $v \in S$, C' has a K_v-rational point lying over $\phi'(q')$.

Remark 6.10 This assumption is valid if any of the following are satisfied:

1. $D \to B$ is unramified at q.
2. $D \to B$ is finite (but perhaps ramified) at q and $L \to B$ has ramification at q of odd order.
3. $D \to B$ is finite (but ramified) at q and $L \to B$ has ramification at q of order two. Choose local uniformizers t, x, and y so that we have local analytic equations $t + ax^2 = 0$ and $t + by^2 = 0$ (with $a, b \in K$) for $D \to B$ and $L \to B$. We assume that ab is a square in K_v.

Note that in the last case, D' and C' have local analytic equations $ax^2 - by^2 = 0$ and $x/y = \pm\sqrt{b/a}$ respectively.

Theorem 6.11 *Under the geometric and arithmetic assumptions made above, $\mathcal{O}_S$-integral points of $(\mathcal{X}, \mathcal{D})$ are Zariski dense.*

Proof. It suffices to prove that $\mathcal{O}_S$-integral points of $(\mathcal{X}', \mathcal{D}')$ are Zariski dense. These map to $\mathcal{O}_S$-integral points $(\mathcal{X}, \mathcal{D})$.

Consider first $\mathcal{O}_S$-integral points of $(\mathcal{L}', \mathfrak{q}')$. These are dense by Proposition 6.5, and contain a finite index subgroup of $\mathbb{G}_a(\mathcal{O}_S) \subset \mathbb{P}^1_K$. Corollary 6.9 and our geometric assumptions imply that infinitely many of these points lie in $\phi'(C'(K_v)) \setminus \phi'(C'(K))$.

Choose a generic $\mathcal{O}_S$-integral point p of $(\mathcal{L}', \mathfrak{q}')$ as described above. Let $\mathcal{X}'_p = \phi'^{-1}(p), \mathcal{D}'_p = \mathcal{X}'_p \cap \mathcal{D}'$, and $\mathcal{L}'_p = \mathcal{X}'_p \cap \mathcal{L}'$, so that $(\mathcal{X}'_p, \mathcal{D}'_p)$ is a rational curve with rational bisection and integral point $\mathcal{L}'_p$. Combining the results of the previous paragraph with Proposition 6.3, with obtain that $\mathcal{O}_S$-integral points of $(\mathcal{X}'_p, \mathcal{D}'_p)$ are Zariski dense. As we vary p, we obtain a Zariski dense collection of integral points for $(\mathcal{X}', \mathcal{D}')$. $\square$

6.5 Cubic surfaces containing a line

Let $\mathcal{X}_1$ be a cubic surface in $\mathbb{P}^3_{\mathcal{O}_S}$, $\mathcal{D}_1 \subset \mathcal{X}_1$ a hyperplane section, and $\mathcal{L}_1 \subset \mathcal{X}_1$ a line not contained in $\mathcal{D}_1$, all assumed to be flat over $\mathrm{Spec}(\mathcal{O}_S)$. Write $\mathfrak{q}_1 := \mathcal{D}_1 \cap \mathcal{L}_1$, a rational section over $\mathrm{Spec}(\mathcal{O}_S)$. Let $\mathbb{P}^3_{\mathcal{O}_S} \dashrightarrow \mathcal{B}$ be the projection associated with $\mathcal{L}_1$, $\mathcal{X} = \mathrm{Bl}_{\mathcal{L}_1}\mathcal{X}_1$, and $\phi : \mathcal{X} \to \mathcal{B}$ the induced projection (of course, $\mathcal{B} = \mathbb{P}^1_{\mathcal{O}_S}$ if $\mathcal{O}_S$ is a UFD). Let $\mathcal{L} \subset \mathcal{X}$ be the proper transform of $\mathcal{L}_1$, $\mathcal{D} \subset \mathcal{X}$ the total transform of $\mathcal{D}_1$, and $\mathfrak{q} = \mathcal{L} \cap \mathcal{D}$. We shall apply Theorem 6.11 to obtain density results for $\mathcal{O}_S$-integral points of $(\mathcal{X}_1, \mathcal{D}_1)$.

We will need to assume the following geometric conditions:

GA1 D_1 is reduced everywhere and nonsingular at q_1;

GA2 X_1 has only rational double points as singularities, with at most one singularity along L_1.

GA3 D_1 is not the union of a line and a conic containing q_1 (defined over K).

Using the first two assumptions, we analyze the projection from the line L_1. This induces a morphism

$$\phi : X \to \mathbb{P}^1.$$

Of course, $X = X_1$ if and only if L_1 is Cartier in X_1, which is the case exactly when X_1 is smooth along L_1. We use L to denote the proper transform of L_1 and D to denote the proper transforms of L_1 and D_1. Our three assumptions imply that D equals the total transform of D_1 and has a unique irreducible component C dominating $\mathbb{P}^1$. We also have that the generic fiber of ϕ is nonsingular, intersects D in two points, and intersects L in two points (if X_1 is smooth along L_1) or in one point (if X_1 has a singularity along L_1). In particular, L is a bisection (resp. section) of ϕ if X_1 is nonsingular (resp. singular) along L_1.

We emphasize that $\mathcal{O}_S$-integral points of $(\mathcal{X}, \mathcal{D})$ map to $\mathcal{O}_S$-integral points of $(\mathcal{X}_1, \mathcal{D}_1)$, and all the Geometric Assumptions of Theorem 6.11 are satisfied except for the last one. The last assumption is verified if any of the following hold:

GA4a The branch loci of $C \to \mathbb{P}^1$ and $L \to \mathbb{P}^1$ do not coincide.

GA4b The curve C has genus one.

GA4c X_1 has a singularity along L_1.

Clearly, either the second or the third condition implies the first.

We turn next to the Arithmetic Assumptions.

AA1 $(\mathcal{L}_1, \mathfrak{q}_1)$ has an $\mathcal{O}_S$-integral point.

Note that $\mathcal{O}_S$-integral points of $(\mathcal{L}_1, \mathfrak{q}_1)$ not lying in the singular locus of $\mathcal{X}_1 \to \mathrm{Spec}(\mathcal{O}_S)$ lift naturally to $\mathcal{O}_S$-integral points of $(\mathcal{L}, \mathfrak{q})$.

Our next task is to translate the conditions of Remark 6.10 to our situation. They are satisfied in any of the following contexts:

AA2a D_1 is irreducible over K and q_1 is not a flex of D_1;

AA2b X_1 has a singularity along L_1;

AA2c D_1 is irreducible over K and q_1 is a flex of D_1. Let H be the hyperplane section containing L_1 and the flex line. We assume that $H \cap X_1 = L_1 \cup M$, where M is a smooth conic.

AA2d D_1 is irreducible over K but q is a flex so that the hyperplane H containing L_1 and the flex line F intersects X_1 in three coincident lines, i.e., $H \cap X_1 = L_1 \cup M_1 \cup M_2$. Choose local coordinates x and y for H so that $L_1 = \{x = 0\}$, $F = \{y = 0\}$, and $M_1 \cup M_2 = \{ax^2 + cxy + by^2 = 0\}$. Then we assume that ab is a square in K_v.

AA2e D_1 consists of a line and a conic C_1 irreducible over K, intersecting in two distinct points, each defined over K_v.

In the first case, the map $D \to B$ is unramified at q. Note that in the second case L is a section for ϕ. In the third case, our assumption implies that $L \to B$ is unramified at q. In the last case, we observe that the points of L lying over $\phi(q)$ are defined over K, hence C' has a K_v-rational point over $\phi'(q')$.

It remains to show that AA2d allows us to apply case 3 of Remark 6.10. We fix projective coordinates on $\mathbb{P}^3$ compatibly with the coordinates already chosen on H: $y = 0$ is the linear equation for the hyperplane containing D_1, $z = 0$ the equation for H, $x = z = 0$ the equations for L_1, and $x = z = w = 0$

the equations for q_1. Under our assumptions, the equations for D_1 and X_1 take the form

$$\begin{aligned} g &:= zw^2 + ax^3 + c_1wxz + c_2wz^2 + c_4x^2z + c_5xz^2 + c_6z^3 = 0 \\ f &:= g + cx^2y + bxy^2 + yz\ell(w,x,y,z) = 0 \end{aligned}$$

where ℓ is linear in the variables. The conic bundle structure $\phi : X \to B$ is obtained by making the substitution $z = tx$

$$\begin{aligned} g' &= tw^2 + x(wc_1t + wc_2t^2) + x^2(a + c_4t + c_5t^2 + c_6t^3) = 0 \\ f' &= g' + cxy + by^2 + ty\ell(w,x,y,tx). \end{aligned}$$

We analyze the local behavior of $D \to B$ at q using x as a coordinate for D. First dehomogenize

$$g'' = t + x(c_1t + c_2t^2) + x^2(a + c_4t + c_5t^2 + c_6t^3) = 0$$

and then take a suitable analytic change of coordinate on D to obtain $t + aX^2 = 0$. To analyze $L \to B$, we set $x = 0$ and use y as a coordinate

$$f'' = t + by^2 + ty\ell(1,0,y,0) = 0.$$

After a suitable analytic change of coordinate on L, we obtain $t + bY^2 = 0$.

Remark 6.12 We further analyze condition AA2d when $K_v = \mathbb{R}$. Then ab is a square if and only if $ab \geq 0$. This is necessarily the case if $c^2 - 4ab < 0$, i.e., if the lines M_1 and M_2 are defined over an imaginary quadratic extension.

We summarize our discussion in the following theorem:

Theorem 6.13 *Let $\mathcal{X}_1$ be a cubic surface, $\mathcal{D}_1 \subset \mathcal{X}_1$ a hyperplane section, and $\mathcal{L}_1 \subset \mathcal{X}_1$ a line not contained in $\mathcal{D}_1$, all assumed to be flat over* $\mathrm{Spec}(\mathcal{O}_S)$. *Write $\mathfrak{q}_1 := \mathcal{D}_1 \cap \mathcal{L}_1$. Assume the following:*

1. *GA1,GA2,GA3, and AA1;*
2. *at least one of the assumptions GA4a,GA4b,or GA4c;*
3. *at least one of the assumptions AA2a,AA2b,AA2c,AA2d, or AA2e.*

Then $\mathcal{O}_S$-integral points of $(\mathcal{X}_1, \mathcal{D}_1)$ are Zariski dense.

We recover the following result (essentially Theorem 2 of Beukers [2]):

Corollary 6.14 Let $\mathcal{X}_1$ be a cubic surface, $\mathcal{D}_1 \subset \mathcal{X}_1$ a hyperplane section, and $\mathcal{L}_1 \subset \mathcal{X}_1$ a line not contained in $\mathcal{D}_1$, all assumed to be flat over $\mathrm{Spec}(\mathbb{Z})$. Write $\mathfrak{q}_1 := \mathcal{D}_1 \cap \mathcal{L}_1$. Assume that

1. X_1 and D_1 are smooth;

2. there exists an $\mathbb{Z}$-integral point of $(\mathcal{L}_1, \mathfrak{q}_1)$;

3. if q is a flex of D_1, we assume that the hyperplane containing L_1 and the flex line intersects X_1 in a smooth conic and L_1.

Then $\mathbb{Z}$-integral points of $(\mathcal{X}_1, \mathcal{D}_1)$ are Zariski dense.

We also recover a weak version of Theorem 1 of [2]. (This theorem is asserted to be true but the proof is not quite complete; the problem occurs in the argument for the second part of Lemma 2.)

Corollary 6.15 Retain all the hypotheses of Corollary 6.14, except that we allow the existence of a hyperplane H intersecting X_1 in three lines L_1, M_1, and M_2 and containing a flex line F for D_1 at q. Let p be a place for $\mathbb{Z}$ (either infinite or finite). Choose local coordinates x and y for H so that $L_1 = \{x = 0\}, F = \{y = 0\}$, and $M_1 \cup M_2 = \{ax^2 + cxy + by^2 = 0\}$, and assume that ab is a square in $\mathbb{Q}_p$. Then $\mathbb{Z}[1/p]$-integral points of $(\mathcal{X}_1, \mathcal{D}_1)$ are Zariski dense (where $\mathbb{Z}[1/\infty] = \mathbb{Z}$ and $\mathbb{Q}_\infty = \mathbb{R}$.)

Of course, there are infinitely many primes p such that ab is a square in $\mathbb{Q}_p$. When $p = \infty$, by Remark 6.12 it suffices to verify that M_1 and M_2 are defined over an imaginary quadratic extension.

We also obtain results in cases where the boundary is reducible:

Corollary 6.16 Let $\mathcal{X}_1$ be a cubic surface, $\mathcal{D}_1 \subset \mathcal{X}_1$ a hyperplane section, and $\mathcal{L}_1 \subset \mathcal{X}_1$ a line not contained in $\mathcal{D}_1$, all assumed to be flat over $\mathrm{Spec}(\mathbb{Z})$. Write $\mathfrak{q}_1 := \mathcal{D}_1 \cap \mathcal{L}_1$. Assume that

1. X_1 is smooth;

2. there exists an $\mathcal{O}_S$-integral point of $(\mathcal{L}_1, \mathfrak{q}_1)$;

3. $D_1 = E \cup C$, where E is a line intersecting L_1 and C is a conic irreducible over K;

4. C intersects E in two points, defined over K_v where v is some place in S;

5. there exists at most one conic in X_1 tangent to both L_1 and C.

Then $\mathcal{O}_S$-integral points of $(\mathcal{X}, \mathcal{D})$ are Zariski dense.

Note that the assumption on the conics tangent to L_1 and C is used to verify GA4a.

6.6 Other applications

Theorem 6.11 can be applied in many situations. We give one further example:

Theorem 6.17 *Let $\mathcal{X} = \mathbb{P}^1_{\mathcal{O}_S} \times \mathbb{P}^1_{\mathcal{O}_S}$, $\mathcal{D} \subset \mathcal{X}$ a divisor of type $(2,2)$, and $\mathcal{L} \subset \mathcal{X}$ a ruling of $\mathcal{X}$, all flat over $\mathcal{O}_S$. Assume that*

1. D is nonsingular;

2. L is tangent to D at q;

3. $\mathcal{O}_S$-integral points of $(\mathcal{L}, \mathfrak{q})$ are Zariski dense.

Then $\mathcal{O}_S$-integral points of $(\mathcal{X}, \mathcal{D})$ are Zariski dense.

Proof. Let ϕ be the projection for which $\mathcal{L}$ is a section. Since $\mathcal{C} = \mathcal{D}$ in this case, the second arithmetic assumption of Theorem 6.11 is easily satisfied. □

7 Potential density for log K3 surfaces

We consider the following general situation:

Problem 7.1 (Integral points of log K3 surfaces) Let X be a surface and D a reduced effective Weil divisor such that (X, D) has log terminal singularities and $K_X + D$ is trivial. Are integral points on (X, D) potentially dense?

Problem 7.1 has been studied when $D = \emptyset$ (see, for example, [3]). In this case density holds if X has infinite automorphisms or an elliptic fibration.

The case $X = \mathbb{P}^2$ and D a plane cubic has also attracted significant attention. Silverman [28] proved potential density in the case where D is singular and raised the general case as an open question. Beukers [1] established this by considering the cubic surface X_1 obtained as the triple cover of X totally branched over D.

Implicit in [2] is a proof of potential density when X_1 is a smooth cubic surface and D_1 is a smooth hyperplane section. Note that this also follows from Theorem 6.13 (cf. also Corollaries 6.14 and 6.15.) After suitable extensions of K and additions to S, there exists a line $L \subset X$ defined over K and the relevant arithmetic assumptions are satisfied. Similarly, the case of $X = \mathbb{P}^1 \times \mathbb{P}^1$ and D a smooth divisor of type $(2, 2)$ follows from Theorem 6.17.

More generally, let X be a smooth Del Pezzo surface of index one, i.e., $K_X\mathbb{Z}$ is saturated in $\mathrm{Pic}(X)$, and with degree $d := K_X^2 \geq 4$. Let D be a smooth anticanonical divisor. Choose general points $W = \{x_1, \ldots, x_{d-3}\} \subset X \setminus D$, and let $X_1 = \mathrm{Bl}_W X$ and D_1 be the proper transform of D. Hence X_1 is a cubic surface, D_1 is a smooth hyperplane section, and the induced map of pairs

$$(X_1, D_1) \to (X, D)$$

is dominant. Since integral points for (X_1, D_1) are potentially dense, the same holds true for (X, D).

We summarize our results as follows:

Theorem 7.2 *Let X be a smooth Del Pezzo surface of degree ≥ 3 and D a smooth anticanonical divisor. Then integral points for (X, D) are potentially dense.*

We close this section with a list of open special cases of Problem 7.1.

1. Let X be a Del Pezzo surface of degree one or two and D an anticanonical cycle. Show that integral points for (X, D) potentially dense.

2. Let X be a Hirzebruch surface and D an anticanonical cycle. Find a smooth rational curve L, intersecting D in exactly one point p, so that the induced map $\varphi : L \to \mathbb{P}^1$ is finite surjective.

7.1 Appendix: some geometric remarks

The reader will observe that the methods employed to prove density for integral points on conic bundles (with bisection removed) are not quite analogous to the methods used for elliptic fibrations. The discrepancy can be seen in a number of ways. First, given a multisection M for a conic bundle (with bisection removed), we can pull-back the conic bundle to the multisection. The resulting fibration has two *rational* sections, Id and τ_M (see section 4). However, *a priori* one cannot control how τ_M intersects the boundary divisor (clearly, this is irrelevant if the boundary is empty). A second explanation may be found in the lack of a good theory of (finite type) Néron models for algebraic tori (see chapter 10 of [5]).

We should remark that in some special cases these difficulties can be overcome, so that integral points may be obtained by geometric methods completely analogous to those used for rational points. Consider the cubic surface

$$x^3 + y^3 + z^3 = 1$$

with distinguished hyperplane at infinity. This surface contains a line with equations $x + y = z - 1 = 0$. Euler showed that the resulting conic bundle admits a multisection $(x_0, y_0, z_0) = (9t^4, 3t - 9t^4, 1 - 9t^3)$, which may be reparametrized as $(x_1, y_1, z_1) = (9t^4, -3t - 9t^4, 1 + 9t^3)$. Lehmer [14] showed that this is the first in a sequence of multisections, given recursively by

$$\begin{aligned}(x_{n+1}, y_{n+1}, z_{n+1}) &= 2(216t^6 - 1)(x_n, y_n, z_n) - (x_{n-1}, y_{n-1}, z_{n-1}) \\ &+ (-108t^4, -108t^4, 216t^4 + 4)\end{aligned}$$

This should be related to the fact that the norm group scheme

$$u^2 - 3(108t^6 - 1)v^2 = 1,$$

admits a section of infinite order $(u, v) = (216t^6 - 1, 12t^3)$.

References

[1] F. Beukers, *Ternary forms equations*, J. Number Theory 54 (1995), no. 1, 113–133.

[2] F. Beukers, *Integral points on cubic surfaces*, Number theory (Ottawa, ON, 1996), 25–33, CRM Proc. Lecture Notes, 19, AMS, Providence, RI, (1999).

[3] F. Bogomolov, Yu. Tschinkel, *Density of rational points on elliptic K3 surfaces*, `alg-geom 9902092`, to appear in Asian J. of Math.

[4] E. Bombieri, D. Masser, U. Zannier, *Intersecting a Curve with Algebraic Subgroups of Multiplicative Groups,* Internat. Math. Res. Notices no. 20 (1999), 1119–1140.

[5] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 21, Springer-Verlag, Berlin, 1990.

[6] J.-L. Colliot-Thélène, A. N. Skorobogatov, P. Swinnerton-Dyer, *Rational points and zero-cycles on fibred varieties: Schinzel's hypothesis and Salberger's device*, J. Reine Angew. Math. 495 (1998), 1–28.

[7] J.-L. Colliot-Thélène, A. N. Skorobogatov, P. Swinnerton-Dyer, *Double fibres and double covers: paucity of rational points*, Acta Arith. 79 (1997), no. 2, 113–135.

[8] J.-L. Colliot-Thélène, A. N. Skorobogatov, P. Swinnerton-Dyer, *Hasse principle and weak approximation for pencils of Severi-Brauer and similar varieties*, J. Reine Angew. Math. 453 (1994), 49–112.

[9] G. Faltings, *Diophantine approximation on abelian varieties*, Ann. of Math. (2) 133 (1991), no. 3, 549–576.

[10] J. Harris, Yu. Tschinkel, *Rational points on quartics*, `alg-geom 9809015`, to appear in Duke Math. J..

[11] B. Hassett, Yu. Tschinkel, *Abelian fibrations and rational points on symmetric products*, `alg-geom 9909074`, (1999).

[12] S. Iitaka, *Algebraic geometry: An introduction to birational geometry of algebraic varieties,* Springer-Verlag, New York-Berlin, 1982.

[13] Y. Kawamata, K. Matsuda, K. Matsuki, *Introduction to the minimal model problem,* Algebraic geometry, Sendai, 1985, 283–360, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam-New York, 1987.

[14] D. H. Lehmer, *On the diophantine equation* $x^3 + y^3 + z^3 = 1$, J. Lond. Math. Soc. 31 (1956), 275–280.

[15] M. McQuillan, *Division points on semi-abelian varieties*, Invent. Math. 120 (1995), no. 1, 143–159.

[16] D. W. Masser, *Specializations of finitely generated subgroups of abelian varieties,* Trans. Amer. Math. Soc. 311 (1989), no. 1, 413–424.

[17] D. W. Masser, *Specializations of endomorphism rings of abelian varieties*, Bull. Soc. Math. France 124 (1996), no. 3, 457–476.

[18] J.S. Milne, *Étale cohomology,* Princeton University Press, Princeton New Jersey, 1980.

[19] D. Mumford, J. Fogarty, F. Kirwan, *Geometric invariant theory* Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (2), 34. Springer-Verlag, Berlin, 1994.

[20] R. Noot, *Abelian varieties—Galois representation and properties of ordinary reduction,* Compositio Math. 97 (1995), no. 1, 161–171.

[21] T. Ono, *On some arithmetic properties of linear algebraic groups,* Ann. of Math. (2) 70 (1959), 266–290.

[22] T. Ono, *Arithmetic of algebraic tori,* Ann. of Math. (2) 74 (1961), 101–139.

[23] J. P. Serre, *Algebraic groups and class fields,* Graduate Texts in Mathematics, 117, Springer-Verlag, New York-Berlin, 1988.

[24] J. P. Serre, *Local fields,* Graduate Texts in Mathematics, 67, Springer-Verlag, New York-Berlin, 1979.

[25] J. P. Serre, *Lectures on the Mordell-Weil theorem,* Aspects of Mathematics, E15. Friedr. Vieweg & Sohn, Braunschweig, 1989.

[26] J.M. Shyr, *A generalization of Dirichlet's unit theorem,* J. Number Theory 9 (1977), no. 2, 213–217.

[27] J. Silverman, *Heights and the specialization map for families of abelian varieties*, J. reine und angew. Math. 342 (1983), 197–211.

[28] J. Silverman, *Integral points on curves and surfaces*, Number theory (Ulm, 1987), 202–241, Lecture Notes in Math., 1380, Springer, New York-Berlin, 1989.

[29] P. Vojta, *Diophantine approximations and value distribution theory*, LNM, 1239, Springer-Verlag, Berlin, 1987.

[30] P. Vojta, *Integral points on subvarieties of semiabelian varieties. II*, Amer. J. Math. 121 (1999), no. 2, 283–313.

[31] V. E. Voskresenskii, *Algebraic groups and their birational invariants,* Translations of Mathematical Monographs, 179. American Mathematical Society, Providence, RI, 1998.